Optical Properties of Solids: Lecture 3

Stefan Zollner
New Mexico State University, Las Cruces, NM, USA
and Institute of Physics, CAS, Prague, CZR (Room 335)
zollner@nmsu.edu or zollner@fzu.cz

These lectures were supported by

- European Union, European Structural and Investment Funds (ESIF)
- Czech Ministry of Education, Youth, and Sports (MEYS), Project IOP Researchers Mobility – CZ.02.2.69/0.0/0.0/0008215

Thanks to Dr. Dejneka and his department at FZU.

EUROPEAN UNION
European Structural and Investment Funds
Operational Programme Research, Development and Education

MINISTRY OF EDUCATION, YOUTH AND SPORTS

NSF: DMR-1505172

http://ellipsometry.nmsu.edu

Optical Properties of Solids: Lecture 3

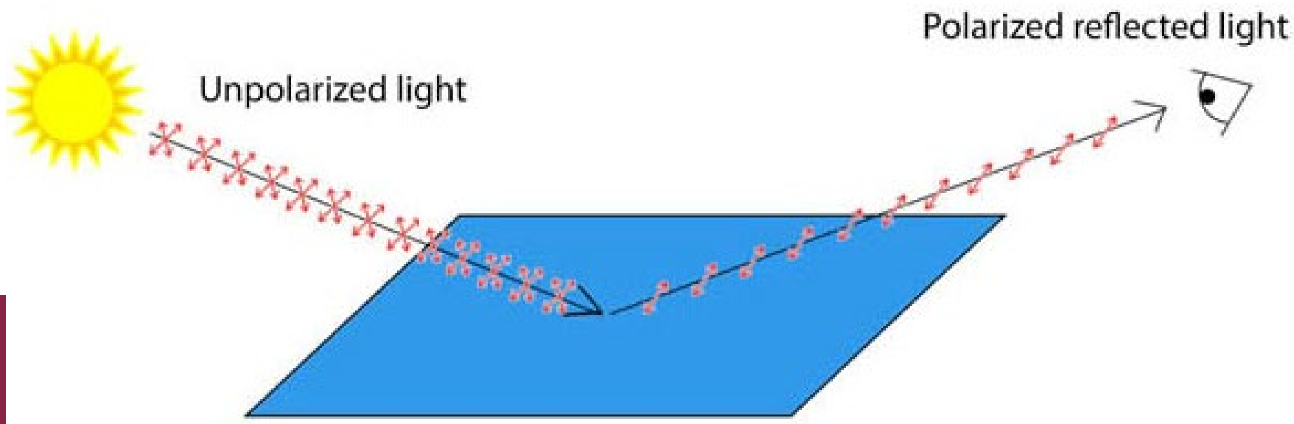
Maxwell's Equations in Vacuum, Plane Waves

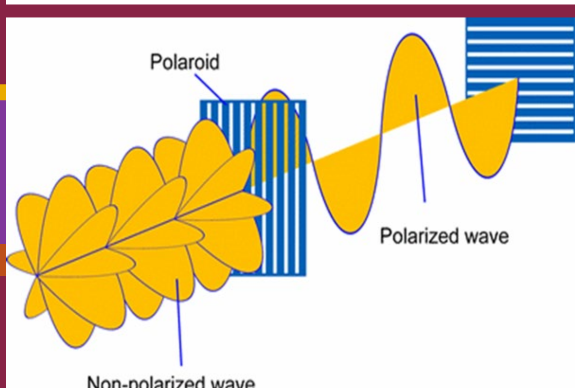
Polarized Light

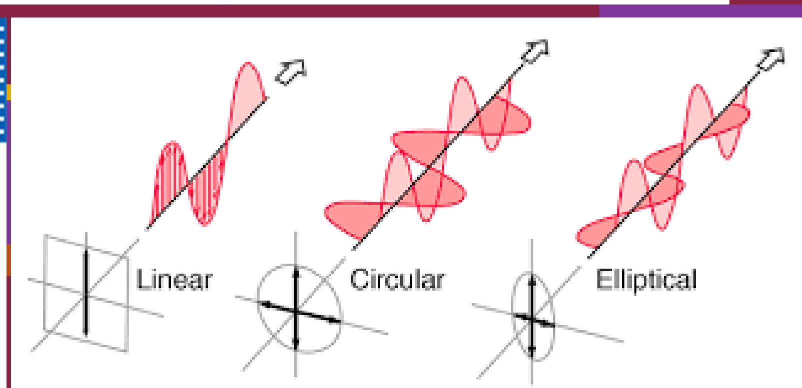
Stokes Parameters, Poincare Sphere

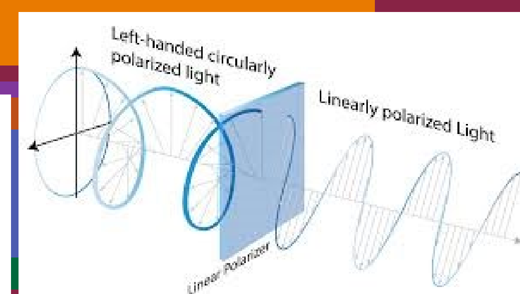
Jones Vectors, Jones Matrix, Mueller Matrix

Decoherence and Depolarization









References: Maxwell's Equations and Ellipsometry

Standard Texts on Electricity and Magnetism:

- J.D. Jackson: *Classical Electrodynamics*
- Landau & Lifshitz, Vol. 2: *Classical Theory of Fields*
- *D.E. Aspnes: **Practical** Electrodynamics (forthcoming)*

Optics:

- E. Hecht: *Optics*
- M. Born, E. Wolf: *Principles of Optics*

Ellipsometry and Polarized Light:

- R.M.A. Azzam and N.M. Bashara: *Ellipsometry and Polarized Light*
- **H.G. Tompkins and E.A. Irene: *Handbook of Ellipsometry* (chapter by Josef Humlicek)**
- H. Fujiwara, *Spectroscopic Ellipsometry*
- H.G. Tompkins and J.N. Hilfiker: *Spectroscopic Ellipsometry*
- H. Fujiwara and R.W. Collins: *Spectroscopic Ellipsometry for PV* (Vol 1+2)
- Zollner: *Propagation of EM Waves in Continuous Media* (Lecture Notes)

Scalars and Vectors

- **Scalar:** Invariant under rotations and inversion
- **Vector:** Transforms like x,y,z under rotation. Sign change under inversion
- **Pseudoscalar:** Invariant under rotations, sign change under inversion
- **Pseudovector:** Transforms like x,y,z under rotation. Invariant under inversion.

BSW	O_h(m-3m)	#	1	4	2_{100}	3	2_{110}	-1	-4	m_{100}	-3	m_{110}	functions	
	Mult.	K	1	6	3	8	6	1	6	3	8	6	.	
Γ_1	A_{1g}	Γ_1^+	1	1	1	1	1	1	1	1	1	1	1, $x^2+y^2+z^2$	Scalar
Γ_1'	A_{1u}	Γ_1^-	1	1	1	1	1	-1	-1	-1	-1	-1	.	Pseudoscalar
Γ_2	A_{2g}	Γ_2^+	1	-1	1	1	-1	1	-1	1	1	-1	.	
Γ_2'	A_{2u}	Γ_2^-	1	-1	1	1	-1	-1	1	-1	-1	1	xyz	
Γ_{12}	E_g	Γ_3^+	2	0	2	-1	0	2	0	2	-1	0	$(2z^2-x^2-y^2, x^2-y^2)$	
Γ_{12}'	E_u	Γ_3^-	2	0	2	-1	0	-2	0	-2	1	0	.	
Γ_{25}	T_{2u}	Γ_5^-	3	-1	-1	0	1	-3	1	1	0	-1	.	
Γ_{25}'	T_{2g}	Γ_5^+	3	-1	-1	0	1	3	-1	-1	0	1	(xy,xz,yz)	
Γ_{15}	T_{1u}	Γ_4^-	3	1	-1	0	-1	-3	-1	1	0	1	(x,y,z)	Vector
Γ_{15}'	T_{1g}	Γ_4^+	3	1	-1	0	-1	3	1	-1	0	-1	(J_x, J_y, J_z)	Pseudovector

Scalar and Vector Waves

- **Field:** Scalar or vector depends on position **r**.
- **Physical quantities are always real**
  Scalar: energy, charge, etc.
  Vector: momentum, current density, electric field, etc.

- Scalar wave

$$s(\vec{r},t) = A\cos\left(\vec{k}\cdot\vec{r} - \omega t + \varphi\right)$$

- Vector wave

$$\vec{E}(\vec{r},t) = \vec{E}_0 A\cos\left(\vec{k}\cdot\vec{r} - \omega t + \varphi\right)$$

- **Where do the complex notations come from?**

Fourier Series of Periodic Functions

- A real-valued scalar function f(t) is called **periodic** with period T, if f(t)=f(t+T) for all values of t.

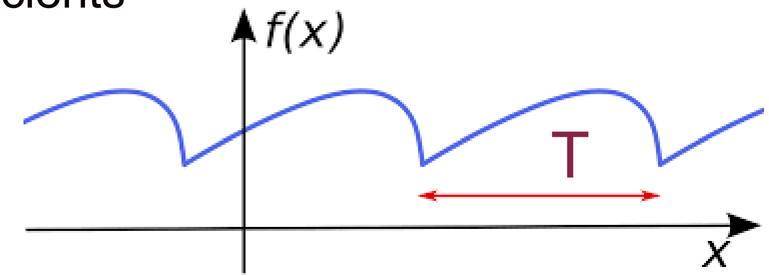
- A periodic scalar function with period T can be written as a **Fourier Series**

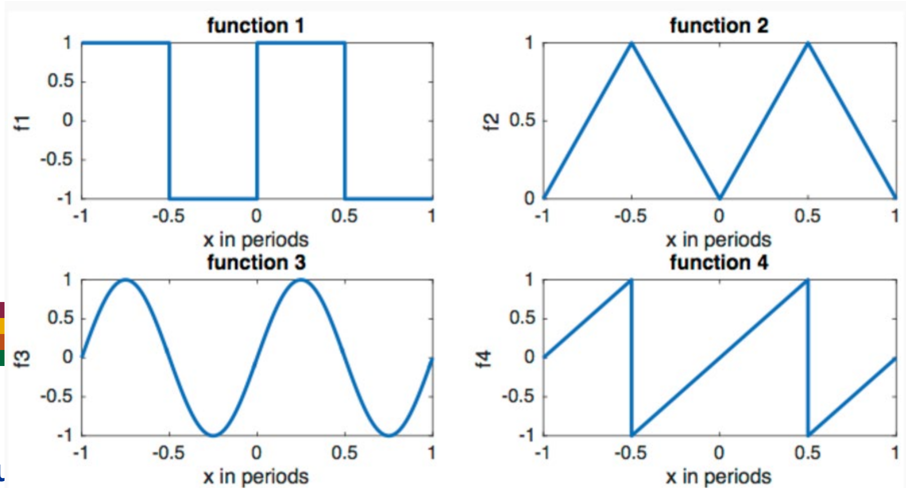
$$f(t) = \frac{1}{2} A_0 + \sum_{m=1}^{\infty} \left[A_m \cos(m\omega t) + B_m \sin(m\omega t) \right]$$

with angular frequency $\omega = 2\pi/T$ and Fourier coefficients

$$A_m = \frac{\omega}{\pi} \int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} f(t) \cos(m\omega t)\, dt$$

$$B_m = \frac{\omega}{\pi} \int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} f(t) \sin(m\omega t)\, dt$$





Jackson, E&M, 1975

Fourier Series of Periodic Functions

- Dealing with harmonic functions (sin, cos) is not convenient, because
  - We need two functions for each harmonic.
  - Taking derivatives is not easy, because sin and cos switch at each order.

- A periodic scalar function with period T can be written as a **Fourier Series**

$$f(t) = \sum_{m=-\infty}^{\infty} c_m \exp(-im\omega t)$$

with **complex** Fourier coefficients

$$c_m = \frac{\omega}{\pi}\int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} f(t)\exp(im\omega t)\,dt = \begin{cases} \frac{A_0}{2} & m = 0 \\ \frac{1}{2}(A_m + iB_m) & m > 0 \\ \frac{1}{2}(A_{-m} - iB_{-m}) & m < 0 \end{cases}$$

- The Fourier coefficients are now complex, but the function f(t) is still real.
- The imaginary parts all cancel, if the complex coefficients c_m are defined correctly.

Jackson, E&M, 1975

Fourier Transforms of Non-Periodic Functions

- If the function f(t) is not periodic, then the period T becomes infinite and the frequency spacing ω between overtones becomes very small.
- The Fourier series now becomes a **Fourier Integral**.

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) \exp(-i\omega t) d\omega$$

with the **Fourier transform F(ω)**

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \exp(i\omega t) dt$$

- The prefactors 1/√2π before the integral can vary (depends on convention).
- **The Fourier transform function F(ω) is allowed to be complex, because it is not a meaningful physical quantity.**
- Orthogonality and completeness:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[i(\omega - \omega')t]\, dt = \delta(\omega - \omega')$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[i\omega(t - t')t]\, d\omega = \delta(t - t')$$

$$\frac{1}{\sqrt{2\pi}} \exp(i\omega t)$$

Orthonormal basis of **Hilbert Space** of real functions

Jackson, E&M, 1975

Math with Fourier Transforms

- Convolution theorem:
  The Fourier transform of a convolution equals the product of the Fourier transforms.

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t')g(t-t')dt'$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (f * g)(t)\exp(i\omega t)dt = \sqrt{2\pi}F(\omega)G(\omega)$$

- The Fourier transform of the derivative of f(t) equals $-i\omega F(\omega)$.

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(t)\exp(i\omega t)dt = -i\omega F(\omega)$$

- The complex conjugate of the Fourier transform equals $F(-\omega)$.

$$\overline{F(\omega)} = F(-\omega)$$

Jackson, E&M, 1975

Fourier Series in Multiple Dimensions

- A real-valued scalar field s(**r**) in a Bravais lattice (with Bravais lattice vectors **T** and reciprocal lattice vectors **G**) is called periodic, if s(**r**+**T**)=s(**r**) for all Bravais lattice vectors **T.**
- A real-valued periodic scalar field s(**r**) in a Bravais lattice can be written as a Fourier sum in reciprocal space

$$s(\vec{r}) = \sum_{\vec{G}} s_{\vec{G}} \exp\left(i\vec{G} \cdot \vec{r}\right)$$

with complex Fourier coefficients

$$s_{\vec{G}} = \frac{1}{V} \int_C s(\vec{r}) \exp\left(-i\vec{G} \cdot \vec{r}\right) d^3\vec{r}$$

where C is the unit cell with volume V. **G** is a reciprocal lattice vector.
- The same equations apply to a real-valued periodic vector field **E**(**r**).

$$\vec{E}(\vec{r}) = \sum_{\vec{G}} \vec{E}_{\vec{G}} \exp\left(i\vec{G} \cdot \vec{r}\right)$$

$$\vec{E}_{\vec{G}} = \frac{1}{V} \int_C \vec{E}(\vec{r}) \exp\left(-i\vec{G} \cdot \vec{r}\right) d^3\vec{r}$$

Ashcroft & Mermin, Appendix D

Fourier Transforms in Multiple Dimensions

- Fourier transforms can also be generalized to multiple dimensions for scalar fields

$$s(\vec{r}) = \left(\frac{1}{\sqrt{2\pi}}\right)^3 \iiint_{-\infty}^{\infty} S\left(\vec{k}\right) \exp\left(i\vec{k}\cdot\vec{r}\right) d^3\vec{k}$$

$$S(k) = \left(\frac{1}{\sqrt{2\pi}}\right)^3 \iiint_{-\infty}^{\infty} s(\vec{r}) \exp\left(-i\vec{k}\cdot\vec{r}\right) d^3\vec{r}$$

- and vector fields

$$\vec{E}(\vec{r}) = \left(\frac{1}{\sqrt{2\pi}}\right)^3 \iiint_{-\infty}^{\infty} \vec{E}\left(\vec{k}\right) \exp\left(i\vec{k}\cdot\vec{r}\right) d^3\vec{k}$$

$$\vec{E}(k) = \left(\frac{1}{\sqrt{2\pi}}\right)^3 \iiint_{-\infty}^{\infty} \vec{E}(\vec{r}) \exp\left(-i\vec{k}\cdot\vec{r}\right) d^3\vec{r}$$

- The fields s(**r**) and **E**(**r**) in real space have real values.
- **The Fourier transforms S(k) and E(k) have complex values, but their imaginary parts cancel out in the summation.**

Microscopic Maxwell's Equations (in Vacuum)

- Electric field strength **E**(**r**)
- Magnetic field strength **H**(**r**)
- Current density **j**(**r**), charge density ρ(**r**)
- Permittivity of free space ε_0, permeability of free space μ_0.

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} = 0$$ Gauss' Law (Coulomb)

$$\vec{\nabla}\cdot\vec{H} = 0$$ Gauss' Law (magnetic field)

$$\vec{\nabla}\times\vec{E} = -\mu_0\frac{\partial\vec{H}}{\partial t}$$ Faraday's Law (Lenz)

$$\vec{\nabla}\times\vec{H} = \vec{j} + \varepsilon_0\frac{\partial\vec{E}}{\partial t} = \varepsilon_0\frac{\partial\vec{E}}{\partial t}$$ Ampere's Law

- Homogeneous (in vacuum), linear, first-order, constant coefficients, partial DEQ.
- Vector analysis can be used (Stokes' Theorem) to transform Maxwell's equations into integral form.
- Introduce speed of light $c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$
- Units: MKSA (SI).

Jackson, E&M, 1975

Wave Equations (in Vacuum)

- Electric field strength **E**(**r**); Magnetic field strength **H**(**r**).
- Maxwell's equations can be combined to obtain the vacuum wave equations (second order, linear, homogeneous, constant coefficients).

$$\vec{\nabla}^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

$$\vec{\nabla}^2\vec{H} - \frac{1}{c^2}\frac{\partial^2\vec{H}}{\partial t^2} = 0$$

$$c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$$

- Plane wave solutions:

$$\vec{E}(\vec{r},t) = \vec{E}_0 \exp\left[i\left(\vec{k}\cdot\vec{r} - \omega t\right)\right]$$

$$\vec{H}(\vec{r},t) = \vec{H}_0 \exp\left[i\left(\vec{k}\cdot\vec{r} - \omega t\right)\right]$$

Why are the solutions complex ?

- Plane wave is not physical (infinite, monochromatic). Form Gaussian wave packets.
- Poynting vector indicates energy flow:

$$\vec{S} = \vec{E} \times \vec{H}$$

Jackson, E&M, 1975

Plane-Wave Solutions to Maxwell's Equations (Vacuum)

- Electric field strength **E**(**r**); Magnetic field strength **H**(**r**).
- Any electric and magnetic field strength can be written as a Fourier-transform

$$\vec{E}(\vec{r},t)=\left(\frac{1}{2\pi}\right)^2\int d\omega\iiint d^3\vec{k}\,\vec{E}\left(\vec{k},\omega\right)\exp\left[i\left(\vec{k}\cdot\vec{r}\right)-\omega t\right]$$

$$\vec{E}\left(\vec{k},\omega\right)=\left(\frac{1}{2\pi}\right)^2\int dt\iiint d^3\vec{r}\,\vec{E}(\vec{r},t)\exp\left[-i\left(\vec{k}\cdot\vec{r}\right)-\omega t\right]$$

- The Fourier transforms are complex, but the **E**(**r**) and **H**(**r**) fields are not.
- Signs: Nebraska convention as modified by Aspnes. Kinetic energy of free particle in quantum mechanics is positive. Classical wave travels along **k**.
- The complex plane waves

$$\vec{E}(\vec{r},t)=\vec{E}_0\exp\left[i\left(\vec{k}\cdot\vec{r}-\omega t\right)\right]$$

$$\vec{H}(\vec{r},t)=\vec{H}_0\exp\left[i\left(\vec{k}\cdot\vec{r}-\omega t\right)\right]$$

are just one term in the Fourier transform. The entire integral is real. (Add complex conjugate.)

- **Solutions to Maxwell's equations are superpositions of plane waves.**

Jackson, E&M, 1975

Fourier-transform Maxwell's Equations

- Substitute plane wave solutions into the differential form of Maxwell's Equations:

$\vec{\nabla} \cdot \vec{E} = 0$ Gauss' Law (Coulomb)

$\vec{\nabla} \cdot \vec{H} = 0$ Gauss' Law (magnetic field)

$\vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$ Faraday's Law

$\vec{\nabla} \times \vec{H} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ Ampere's Law

$$\vec{E}(\vec{r},t) = \vec{E}_0 \exp\left[i\left(\vec{k}\cdot\vec{r} - \omega t\right)\right]$$

$$\vec{H}(\vec{r},t) = \vec{H}_0 \exp\left[i\left(\vec{k}\cdot\vec{r} - \omega t\right)\right]$$

$\vec{k} \cdot \vec{E}_0 = 0$ Gauss' Law (Coulomb)

$\vec{k} \cdot \vec{H}_0 = 0$ Gauss' Law (magnetic field)

$\vec{k} \times \vec{E}_0 = \omega\mu_0\vec{H}_0$ Faraday's Law

$\vec{k} \times \vec{H}_0 = -\omega\varepsilon_0\vec{E}_0$ Ampere's Law

Fourier-transform Maxwell's Equations

$$\vec{E}(\vec{r},t) = \vec{E}_0 \exp\left[i\left(\vec{k}\cdot\vec{r} - \omega t\right)\right]$$

$$\vec{H}(\vec{r},t) = \vec{H}_0 \exp\left[i\left(\vec{k}\cdot\vec{r} - \omega t\right)\right]$$

$\vec{k}\cdot\vec{E}_0 = 0$ Gauss' Law (Coulomb)

$\vec{k}\cdot\vec{H}_0 = 0$ Gauss' Law (magnetic field)

$\vec{k}\times\vec{E}_0 = \omega\mu_0\vec{H}_0$ Faraday's Law

$\vec{k}\times\vec{H}_0 = -\omega\varepsilon_0\vec{E}_0$ Ampere's Law

$k^2 = \frac{\omega^2}{c^2}$ Wave equation (Dispersion relation)

Any solution to Maxwell's equation in vacuum can be written as a superposition of plane waves.

EM waves are transverse (**E**, **H** perpendicular to **k**).

E $\perp$ **H**, $E_0=Z_0H_0$, $Z_0=\sqrt{(\mu_0/\varepsilon_0)}=377\ \Omega$ impedance of vacuum.

Jackson, E&M, 1975

Polarized Light; Jones Vectors

$$\vec{E}(\vec{r},t) = \vec{E}_0 \exp\left[i\left(\vec{k}\cdot\vec{r} - \omega t\right)\right]$$

- Select **k** along the z-axis. Then two field components E_x and E_y are sufficient.

$$\vec{E}(\vec{r},t) = \begin{pmatrix} E_{0x} \\ E_{0y} \end{pmatrix} \exp[i(kz - \omega t)]$$

- An EM wave is described by seven (7) real quantities:
  - Direction of wave vector (two angles ϕ and θ).
  - Magnitude of wave vector (and angular frequency).
  - Two complex amplitudes E_{0x} and E_{0y} (**Jones vector**).
  - One of these (**absolute phase**) cannot be measured; leaving six parameters.

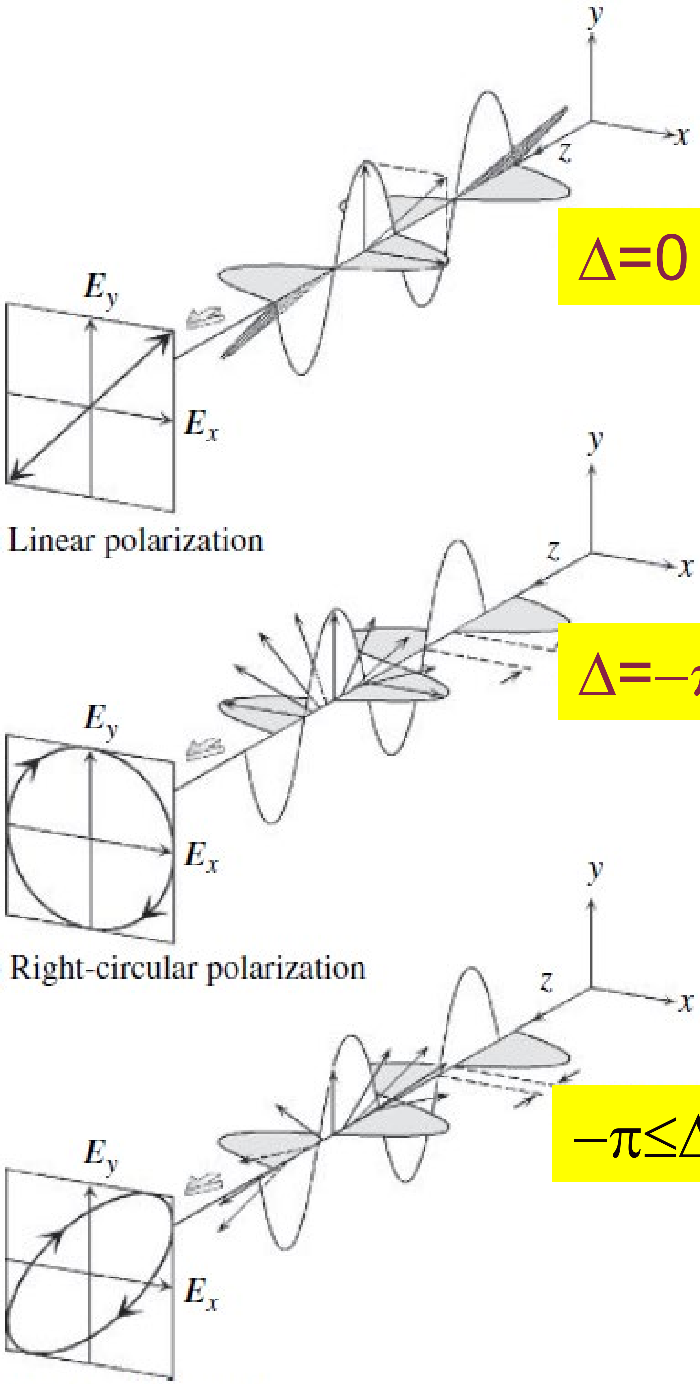
$$\begin{pmatrix} E_{0x} \\ E_{0y} \end{pmatrix} = E_0 \begin{pmatrix} X \exp i\Delta_X \\ Y \exp i\Delta_Y \end{pmatrix} = E_0 \begin{pmatrix} \sin\psi \exp i\Delta \\ \cos\psi \end{pmatrix} \exp i\Delta_y$$

  - We don't care about the **light intensity** and the absolute phase.
  - **ψ and Δ are called the ellipsometric angles; describe polarization of wave.**
  - **ψ=arctan(X/Y); $\Delta=\Delta_X-\Delta_Y$; $\rho=\tan\psi\exp(i\Delta)$;**

J. Humlicek, in Tompkins & Irene (Handbook of Ellipsometry)

NM STATE

Polarized Light; Jones Vectors



(a) Linear polarization

(b) Right-circular polarization

(c) Elliptical polarization

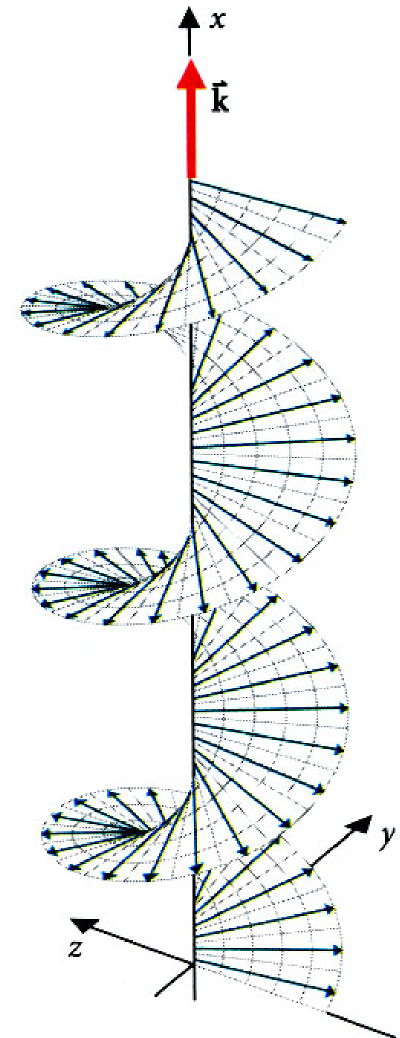
$\Delta=0$

$\Delta=-\pi/2$

$-\pi\leq\Delta\leq\pi$

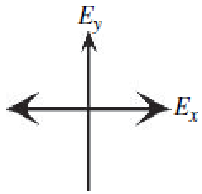
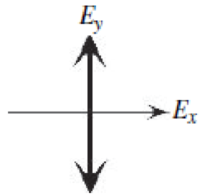
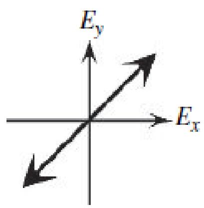
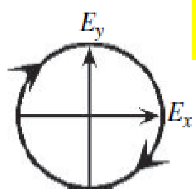
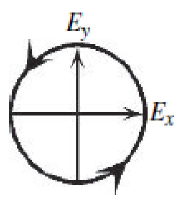
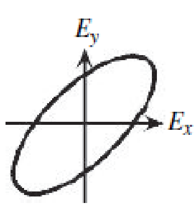
Jones Vector

$$\begin{pmatrix} \sin\psi \exp i\Delta \\ \cos\psi \end{pmatrix}$$



H. Fujiwara

Polarized Light; Jones Vectors

Polarization	Polarization state	Jones vector		
Linear polarization parallel to x axis		$\begin{bmatrix}1\\0\end{bmatrix}$	p-polarized	$\Delta=0$, $\psi=\pi/2$
Linear polarization parallel to y axis		$\begin{bmatrix}0\\1\end{bmatrix}$	s-polarized	$\Delta=0$, $\psi=0$
Linear polarization oriented at 45°		$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$		$\Delta=0$, $\psi=\pi/4$
Right-circular polarization		$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$	right-circular	$\Delta=-\pi/2$, $\psi=\pi/4$
Left-circular polarization		$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$	left-circular	$\Delta=\pi/2$, $\psi=\pi/4$
Elliptical polarization		$\begin{bmatrix}\sin\psi\exp(i\Delta)\\\cos\psi\end{bmatrix}$	elliptical	$-\pi\leq\Delta\leq\pi$, $0\leq\psi\leq\pi/2$

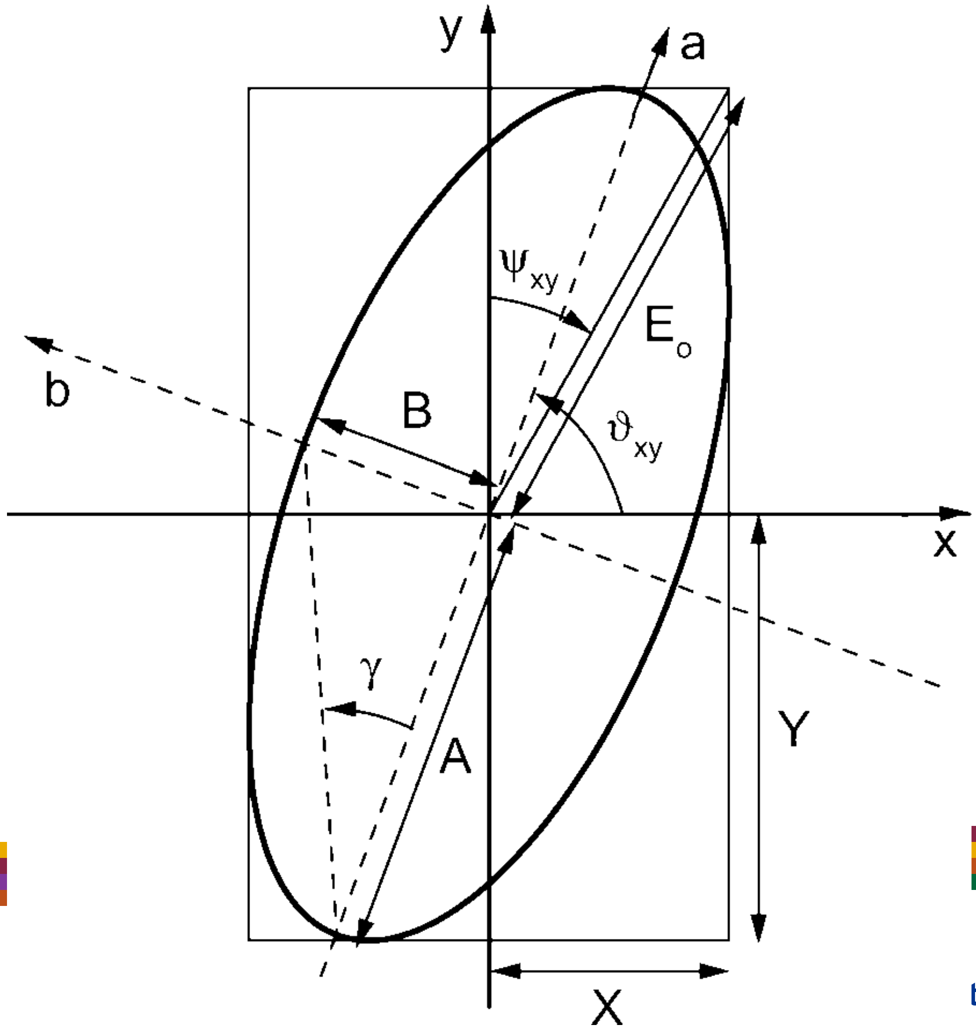
The polarization state of polarized light can be described with two parameters **ψ and Δ** called **ellipsometric angles**.

H. Fujiwara

Polarization Ellipse

$$\vec{E}(z=0,t) = E_0 \begin{pmatrix} \sin\psi \exp i\Delta \\ \cos\psi \end{pmatrix} \exp[-i\omega(t-\tau)t]$$

At z=0, the electric field vector traces out an ellipse.



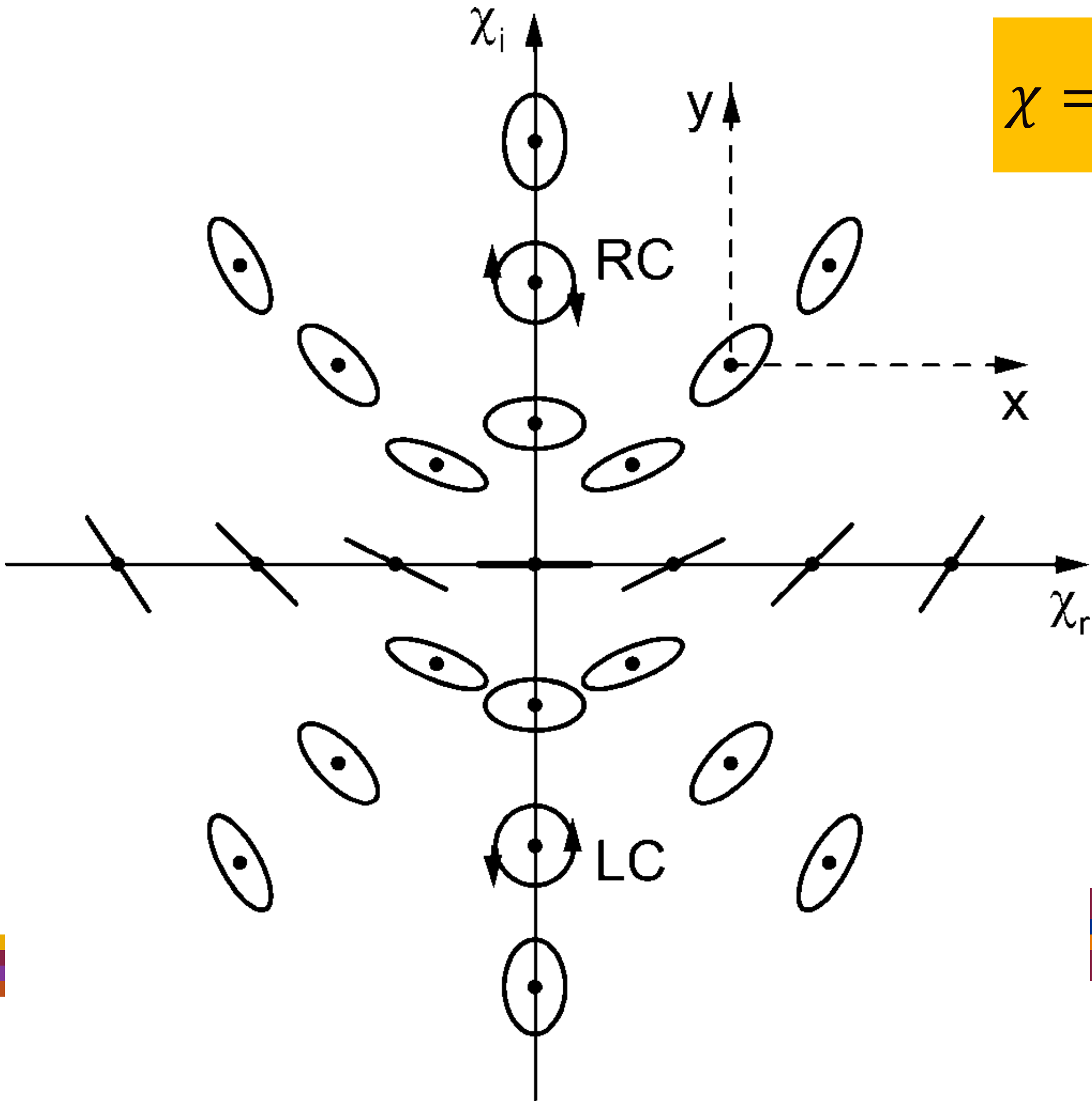
Parameters of the ellipse:

- Azimuth ϑ
- Ratio tanγ major/minor axis
  Ellipticity e=tanγ=B/A

can be calculated from ψ,Δ.

Representation of Polarized Light by Complex Numbers

$$\vec{E}(z=0,t) = E_0 \begin{pmatrix} \sin\psi \exp i\Delta \\ \cos\psi \end{pmatrix} \exp[-i\omega(t-\tau)t]$$



$$\chi = \frac{\exp i\Delta}{\tan\psi} = \frac{\tan\vartheta + i\tan\gamma}{1 - i\tan\vartheta\tan\gamma}$$

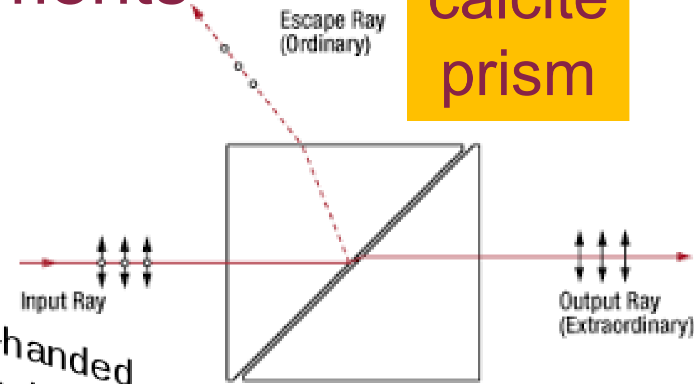
Complex number χ is related to ellipticity and azimuth of the polarization ellipse.

Also **Jones ratio**:

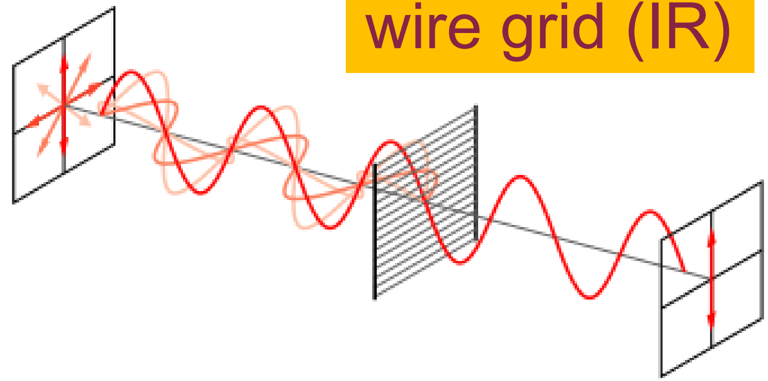
$$\rho = \tan\psi \exp i\Delta = \frac{E_x}{E_y}$$

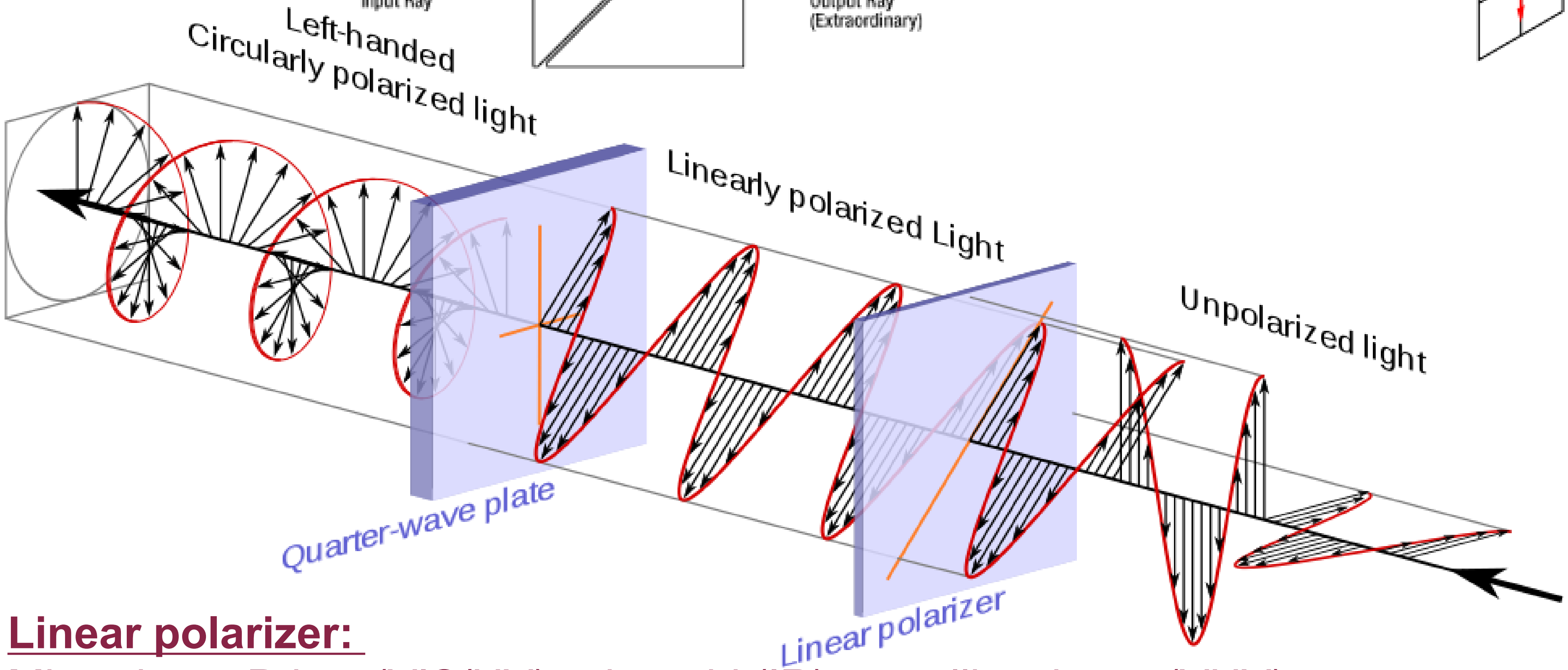
Optical Elements

calcite prism



wire grid (IR)





Linear polarizer:

Mica sheet, Prism (VIS/UV), wire grid (IR), metallic mirrors (VUV)

Compensator/retarder:

Quarter-wave plate

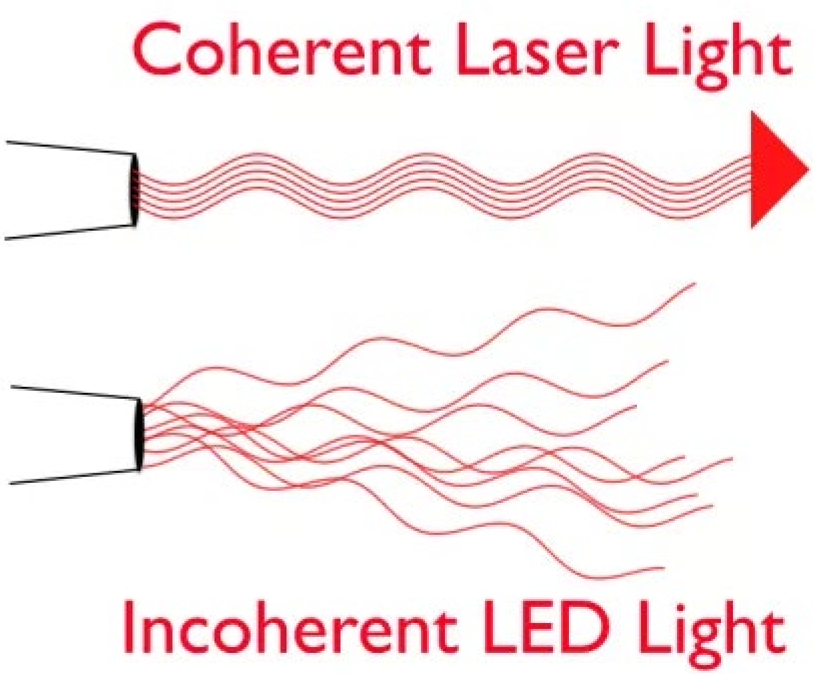
$$\Delta = \frac{2\pi}{\lambda}|n_e - n_o|\mathrm{d} = \frac{\pi}{2}$$

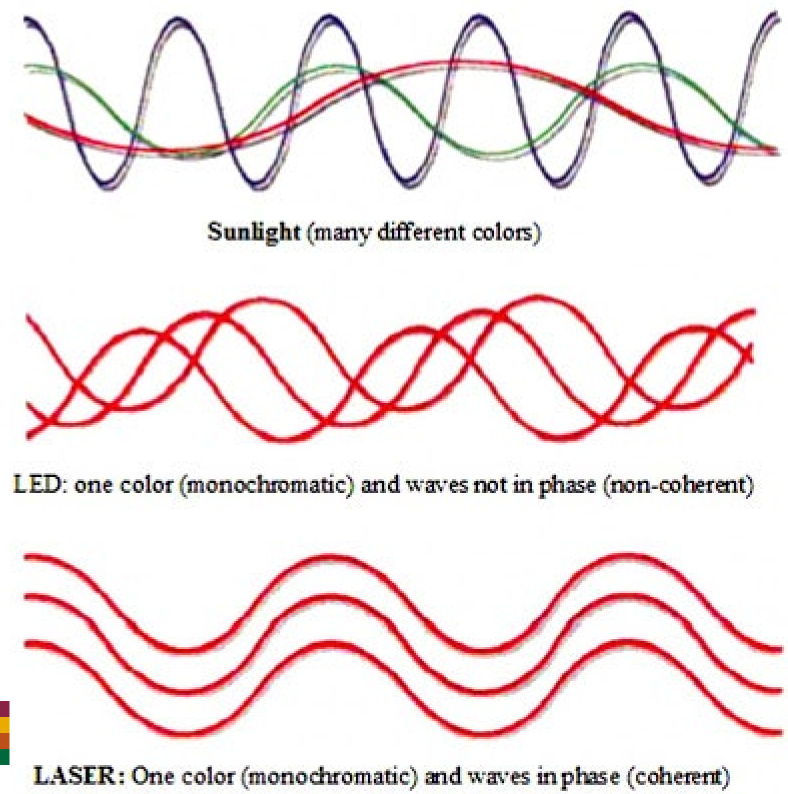
H. Fujiwara

Decoherence and Depolarization

$$\vec{E}(\vec{r},t) = \vec{E}_0 \exp\left[i\left(\vec{k}\cdot\vec{r} - \omega t\right)\right]$$

In practice, light sources are superpositions with several frequencies, called **wave packets**.
Similarly, light sources have **mixed polarization states**.





Stokes Parameters

$$\vec{E}(\vec{r},t) = \vec{E}_0 \exp\left[i\left(\vec{k}\cdot\vec{r} - \omega t\right)\right]$$

$$S_0 = I_x + I_y = E_x E_x^* + E_y E_y^*$$ Total intensity

$$S_1 = I_x - I_y = E_x E_x^* - E_y E_y^*$$ s-polarized minus p-polarized

$$S_2 = I_{45°} - I_{-45°} = E_x E_y^* + E_x^* E_y$$ Diagonal difference

$$S_3 = I_R - I_L = 2\,\mathrm{Re}\left(E_x^* E_y\right)$$ Right minus left circular

$$0 \le p = \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0} \le 1$$

Degree of Polarization (%)

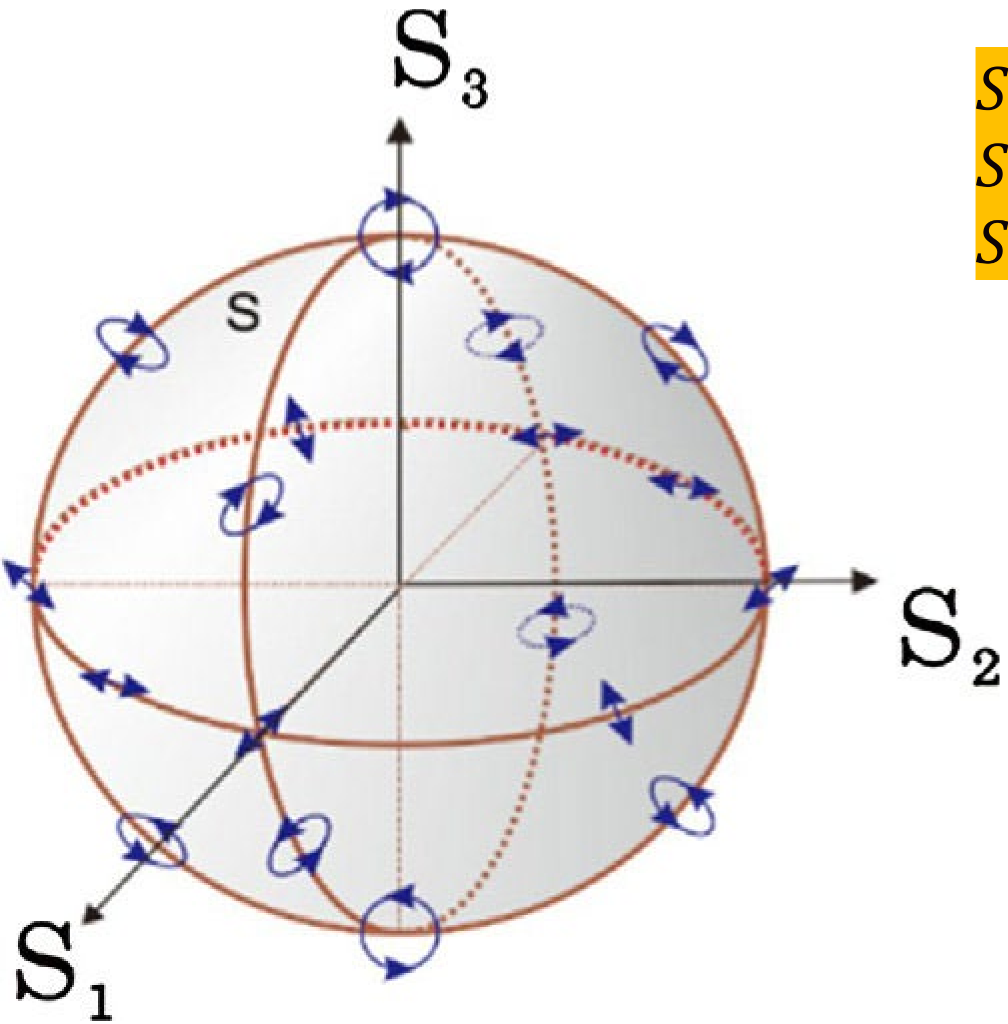
$$\tan(2\vartheta) = \frac{S_2}{S_1}$$

$$\tan(2\gamma) = \frac{S_3}{\sqrt{S_1^2 + S_2^s}}$$

The Stokes parameters are related to the azimuth ϑ and ellipticity γ of the polarization ellipse.

H. Fujiwara

Poincare Sphere



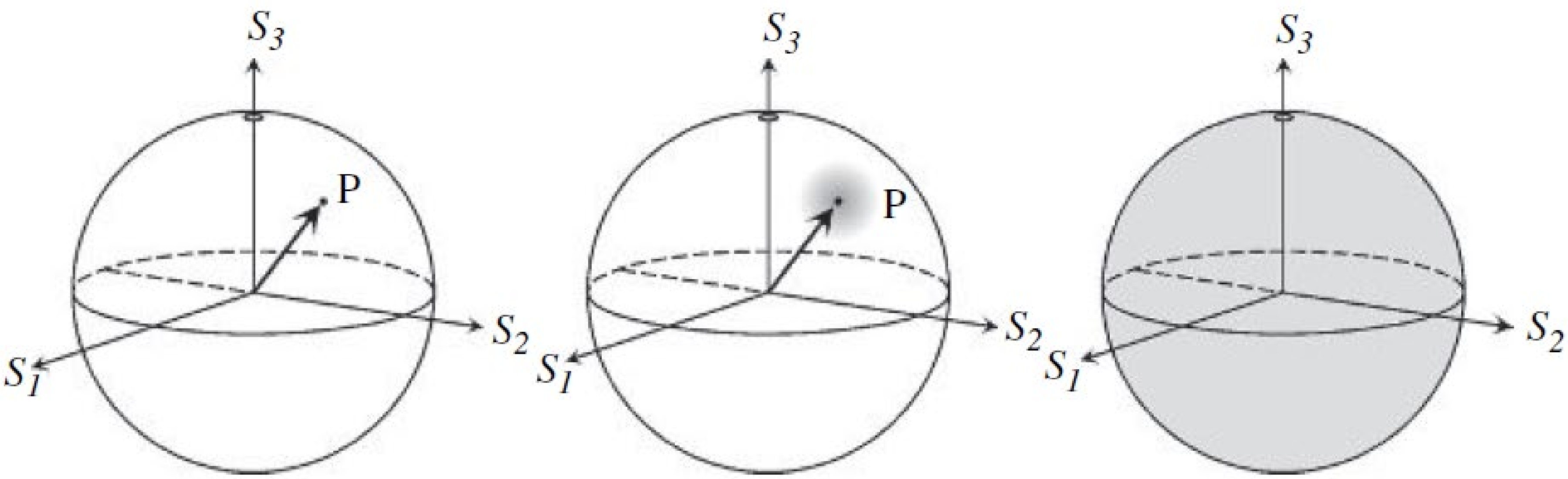
$$S_1 = \cos 2\gamma \cos 2\vartheta = -\cos 2\psi$$
$$S_2 = \cos 2\gamma \sin 2\vartheta = \sin 2\psi \cos \Delta$$
$$S_3 = \sin 2\gamma = -\sin 2\psi \sin \Delta$$

The Stokes parameters for completely polarized light, taken as coordinates, define a point on the surface of a sphere.

Poles: Circularly polarized
Equator: Linearly polarized

H. Fujiwara

Partially Polarized Light



(a) Totally polarized light (b) Partially polarized light (c) Unpolarized light

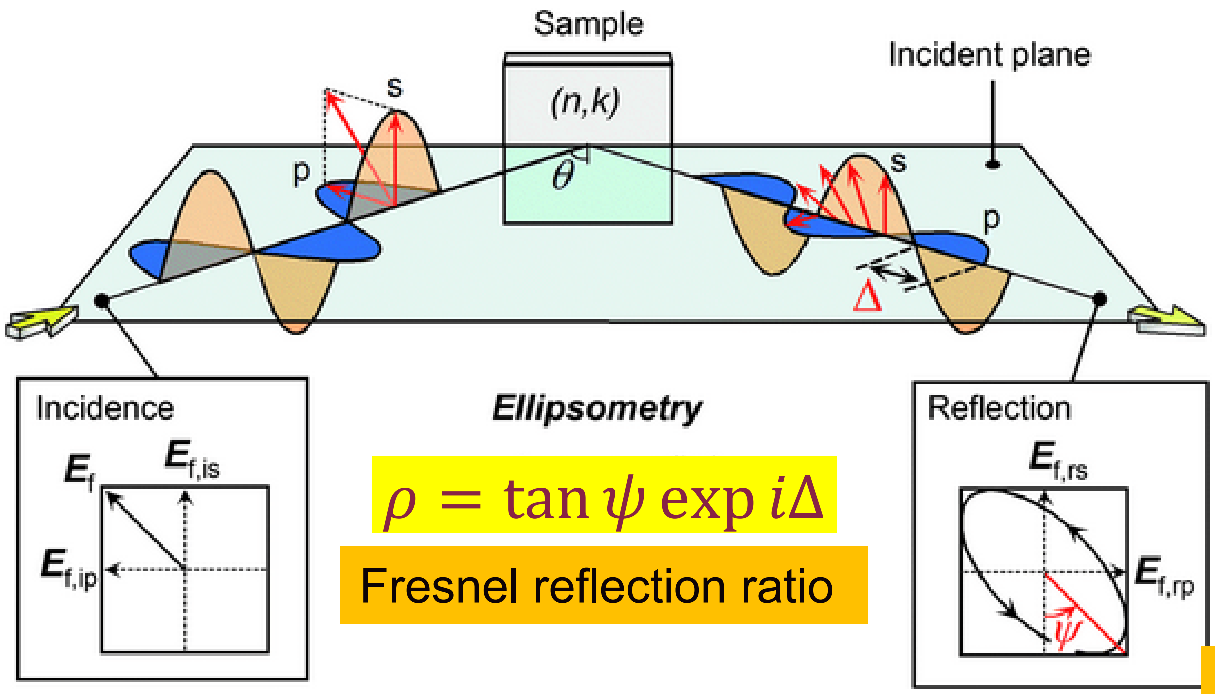
Totally polarized light: point on the surface of Poincare sphere.
Partially polarized light: region in the sphere.
Completely unpolarized light: Point in the center of the sphere.

H. Fujiwara

Jones Matrix

Optical element	Corresponding Jones matrix
Linear polarizer with axis of transmission horizontal[1]	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
Linear polarizer with axis of transmission vertical[1]	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
Linear polarizer with axis of transmission at ±45° with the horizontal[1]	$\frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}$
Quarter-wave plate with fast axis vertical[2][note 1]	$e^{\frac{i\pi}{4}}\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$
Quarter-wave plate with fast axis horizontal[2]	$e^{-\frac{i\pi}{4}}\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$
Half-wave plate with fast axis at angle θ w.r.t the horizontal axis[3]	$e^{-\frac{i\pi}{2}}\begin{pmatrix} \cos^2\theta - \sin^2\theta & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & \sin^2\theta - \cos^2\theta \end{pmatrix}$

Ellipsometry Measurement (simplified)



Polarization State
Jones Vector

$$J = \begin{pmatrix} E_{0x} \\ E_{0y} \end{pmatrix}$$

Ellipsometry Experiment

$$\rho = \tan\psi \exp i\Delta$$

Fresnel reflection ratio

$$J_{\text{out}} = \begin{pmatrix} r_{pp} & r_{ps} \\ r_{sp} & r_{ss} \end{pmatrix} J_{\text{in}}$$

Fresnel reflection coefficients

$$\begin{pmatrix} r_{pp} & r_{ps} \\ r_{sp} & r_{ss} \end{pmatrix} = r_{ss} \begin{pmatrix} \rho & \rho_{ps} \\ \rho_{sp} & 1 \end{pmatrix}$$

Anisotropy or depolarization (not both)

Isotropic surface:
Off-diagonal elements vanish.

$$\begin{pmatrix} r_{pp} & r_{ps} \\ r_{sp} & r_{ss} \end{pmatrix} = r_{ss} \begin{pmatrix} \rho & 0 \\ 0 & 1 \end{pmatrix}$$

Fujiwara & Collins

NM STATE

Ellipsometry: How does it work ?

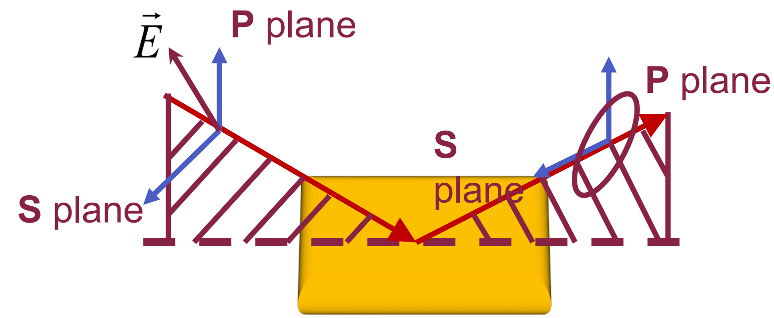
Fresnel Reflectance ratio

$$\rho = \frac{R_p}{R_s} = \frac{E_{rp}}{E_{ip}} \cdot \frac{E_{is}}{E_{rs}} = \tan \Psi e^{i\Delta}$$

$$\tilde{\varepsilon} = \sin^2 \phi \left[1 + \tan^2 \phi \cdot \left(\frac{1-\rho}{1+\rho} \right)^2 \right]$$

Angle of incidence

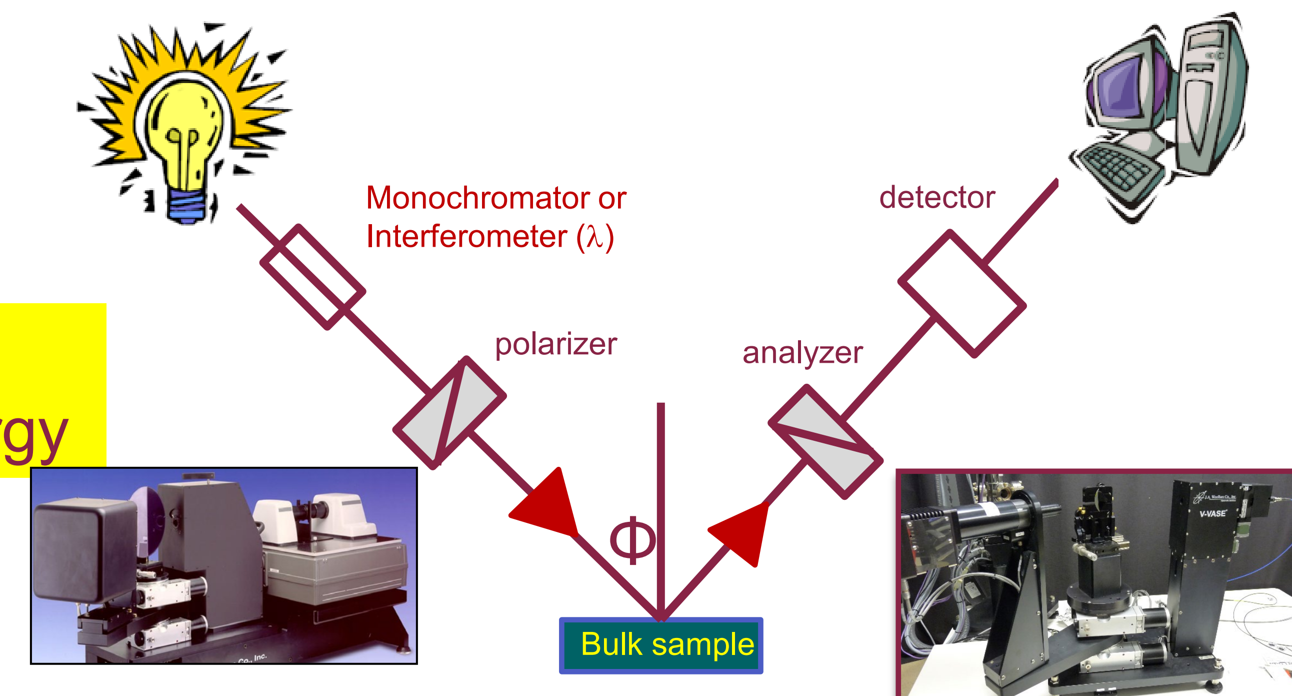
We measure the change in the polarization state of light, when it is reflected by a flat surface (bulk).



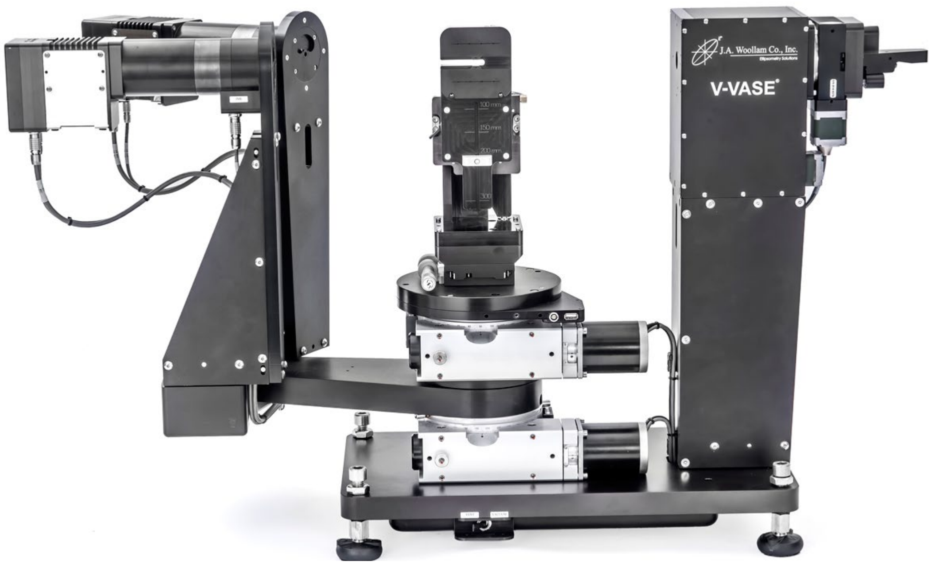
Result:

$$\tilde{\varepsilon} = \varepsilon_1 + i\varepsilon_2$$

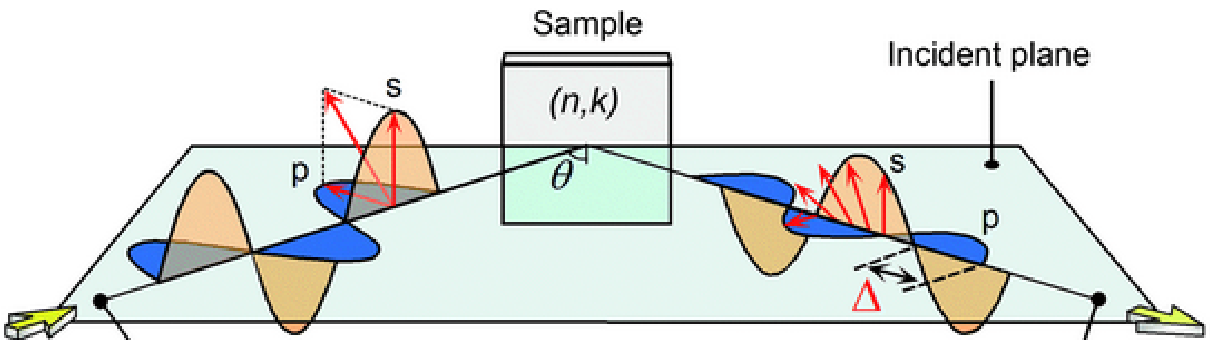
Optical constants versus photon energy

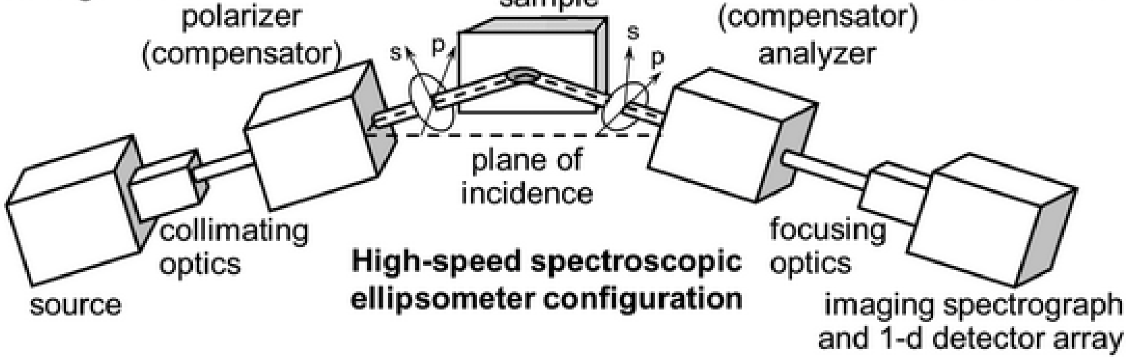


Ellipsometry Instrumentation



Mueller Matrix Ellipsometry





Polarization State
Stokes Vector

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}$$

Ellipsometry Experiment

$$S_{\text{out}} = \hat{M} S_{\text{in}}$$

$$\hat{M} = \begin{pmatrix} M_{00} & M_{01} & M_{02} & M_{03} \\ M_{10} & M_{11} & M_{12} & M_{13} \\ M_{20} & M_{21} & M_{22} & M_{23} \\ M_{30} & M_{31} & M_{32} & M_{33} \end{pmatrix}$$

Measures anisotropy and depolarization
Requires two rotating compensators

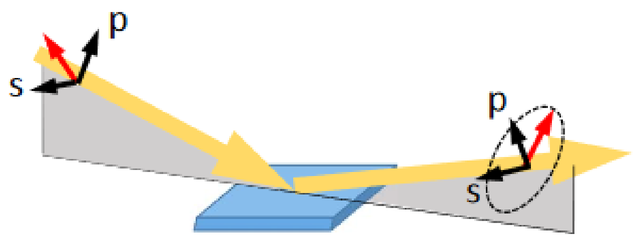
Beetles, cancer research, magnetic field

Fujiwara & Collins

Mueller Matrices

Isotropic sample, no depolarization

$$\mathbf{J}_{sample} = \begin{bmatrix} r_p & 0 \\ 0 & r_s \end{bmatrix}$$



$$\mathbf{M}_{sample} = \begin{bmatrix} 1 & -N & 0 & 0 \\ -N & 1 & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix}$$

$$N = \cos(2\psi)$$
$$S = \sin(2\psi)\sin(\Delta)$$
$$C = \sin(2\psi)\cos(\Delta)$$
$$N^2 + S^2 + C^2 = 1$$

Standard ellipsometry:
- Thickness measurements of thin films
- Optical functions of isotropic materials

$$\rho = (\rho_{real} + i\rho_{imag}) = \frac{r_p}{r_s} = \tan(\psi)e^{i\Delta} = \frac{C + iS}{1 + N}$$

This Mueller matrix depends only on 2 parameters

O. Arteaga

Summary

- Fourier series and Fourier transforms
- Plane waves
- Maxwell's Equations in vacuum (general and plane-wave format)
- Polarized light
  - Jones and Stokes vectors
  - Polarization ellipse
  - Poincare sphere
- Ellipsometry experiments
  - Jones and Mueller matrices